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# On the distribution of a second-class particle in the asymmetric simple exclusion process

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**Abstract**
We give an exact expression for the distribution of the position $X(t)$ of a single second-class particle in the asymmetric simple exclusion process (ASEP) where initially the second-class particle is located at the origin and the first-class particles occupy the sites $\mathbb{Z}^+ = \{1, 2, \ldots\}$.

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## 1. Introduction

The asymmetric simple exclusion process (ASEP) [2, 3] is one of the simplest models of nonequilibrium statistical mechanics and has been called the 'default stochastic model for transport phenomena' [8]. A useful concept in exclusion processes is that of a *second-class particle.*[3]

> Imagine that the particles in the system are each called either first class or second class. The evolution is the same as before, except that if a second-class particle attempts to go to a site occupied by a first-class particle, it is not allowed to do so, while if a first-class particle attempts to move to a site occupied by a second-class particle, the two particles exchange positions. In other words, a first-class particle has priority over a second-class particle. This rule has no effect on whether or not a given site is occupied at a given time. The advantage, though, is that viewed by itself, the collection of first-class particles is Markovian, and has the same law as the exclusion process. The collection of second-class particles is clearly not Markovian. However, the collection of first- and second-class particles is Markovian, and again evolves like an exclusion process.

[3] The following quote is taken from Liggett [3].

Here we consider ASEP on the integer lattice $\mathbb{Z}$ with jumps one step to the right with rate $p$ and jumps one step to the left with rate $q = 1 - p$. We assume a leftward drift, i.e. $q > p$. We further assume that the system has one second-class particle initially located at the origin and first-class particles initially located at sites in

$$Y = \{0 < y_1 < y_2 < \cdots\} \subset \mathbb{Z}^+.$$

With the above initial condition, we denote by $X(t)$ the position of the second-class particle at time $t$. The purpose of this note is to give an exact expression for the probability that the second-class particle is at position $x$ at time $t$, i.e. $\mathbb{P}_Y(X(t) = x)$. (The subscript $Y$ denotes the sites of the initial configuration of the first-class particles.) Our main result is for $Y = \mathbb{Z}^+$ and is given below in (9) and in a slightly different form in (11).

## 2. A basic lemma

The single second-class particle located at $X(t)$ can be viewed as the (single) discrepancy under *basic coupling* between two asymmetric simple exclusion processes $\eta_t$ and $\zeta_t$, where $\zeta_t(X(t)) = 1$ and $\eta_t(X(t)) = 0$ and initially $\{x : \zeta_0(x) = 1\} = Y' = \{0\} \cup Y$ and $\{x : \eta_0(x) = 1\} = Y$ [2, 3].

We first learned the following identity from H Spohn [5] but presumably it has a long history:

$$\mathbb{P}_Y(X(t) = x) = \mathbb{P}_{Y'}(\zeta_t(x) = 1) - \mathbb{P}_Y(\eta_t(x) = 1)\,. \tag{1}$$

For the convenience of the reader, we give a short proof of (1). Let $\zeta_t$ and $\eta_t$ be as above evolving together under the basic coupling [2, 3]. Recall that the coupled processes satisfy $\eta_t \leqslant \zeta_t$ for all $t > 0$ since they satisfy this inequality at $t = 0$ [2, 3].[4] Define

$$\mathcal{J}_\eta(x,t) := \sum_{z\leqslant x} \eta_t(z) = \text{number of particles in configuration } \eta_t \text{ with positions } \leqslant x,$$

$$\mathcal{J}_\zeta(x,t) := \sum_{z\leqslant x} \zeta_t(z) = \text{number of particles in configuration } \zeta_t \text{ with positions } \leqslant x,$$

$$\mathcal{I}(x,t) = \begin{cases} 1 & \text{if} \quad X(t) \leqslant x, \\ 0 & \text{if} \quad X(t) > x. \end{cases}$$

By counting

$$\mathcal{J}_\zeta(x,t) = \mathcal{J}_\eta(x,t) + \mathcal{I}(x,t). \tag{2}$$

Since

$$\mathbb{E}_{Y'}(\mathcal{J}_\zeta(x,t)) = \sum_{z\leqslant x} \mathbb{E}_{Y'}(\zeta_t(z)) = \sum_{z\leqslant x} \mathbb{P}_{Y'}(\zeta_t(z) = 1),$$

$$\mathbb{E}_{Y}(\mathcal{J}_\eta(x,t)) = \sum_{z\leqslant x} \mathbb{E}_{Y}(\eta_t(z)) = \sum_{z\leqslant x} \mathbb{P}_{Y}(\eta_t(z) = 1),$$

$$\mathbb{E}_{Y}(\mathcal{I}(x,t)) = \mathbb{P}_Y(X(t) \leqslant x) = \sum_{z\leqslant x} \mathbb{P}_Y(X(t) = z),$$

the expectation of (2) gives

$$\sum_{z\leqslant x} \mathbb{P}(X(t) = z) = \sum_{z\leqslant x} \mathbb{P}(\zeta_t(z) = 1) - \sum_{z\leqslant x} \mathbb{P}(\eta_t(z) = 1)$$

from which (1) follows.

[4] Given two configurations $\eta, \zeta \in \{0,1\}^{\mathbb{Z}}$ we say $\eta \leqslant \zeta$ if $\eta(x) \leqslant \zeta(x)$ for all $x \in \mathbb{Z}$.

## 3. Probability for a site to be occupied in ASEP

For ASEP with particles initially at $Y$ we denote by $x_m(t)$ the position of the $m$th left-most particle at time $t$ (so $x_m(0) = y_m$). In theorem 5.2 of [6] the authors gave an exact expression for $\mathbb{P}_Y(x_m(t) = x)$. To state this result we first recall the definition of the $\tau$-binomial coefficients. For $0 \leqslant \tau := p/q < 1$ we define for each $n \in \mathbb{Z}^+$

$$[n] = \frac{1-\tau^n}{1-\tau}, \qquad [n]! = [n][n-1]\cdots[1], \qquad [0]! := 1,$$
$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!}, \qquad 0 \leqslant k \leqslant n,$$

and if $k > n$ we set $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right] = 0$. Equation (5.12) of [6] can be written in the following way:[5,6]

$$\mathbb{P}_Y(x_m(t) = x) = \sum_{k=1}^{|Y|} \sum_{\substack{S\subset Y \\ |S|=k}} c_{m,k} \tau^{\sigma(S,Y)} \int_{\mathcal{C}_R} \cdots \int_{\mathcal{C}_R} I(x,k,\xi) \prod_{i=1}^{k} \xi_i^{-s_i} \, \mathrm{d}^k\xi, \tag{3}$$

where, if $S := \{s_1, \ldots, s_k\}$ then

$$c_{m,k} = q^{k(k-1)/2}(-1)^{m+1}\tau^{m(m-1)/2}\tau^{-km}\begin{bmatrix} k-1 \\ k-m \end{bmatrix},$$
$$\sigma(S,Y) = \#\{(s,y) : s \in S,\, y \in Y, \text{ and } y \leqslant s\}$$
$$= \text{sum of the positions of the elements of } S \text{ in } Y,$$
$$I(x,k,\xi) = \prod_{1\leqslant i<j\leqslant k} \frac{\xi_j - \xi_i}{p + q\xi_i\xi_j - \xi_i}\left(1 - \prod_{i=1}^{k}\xi_i\right)\prod_{i=1}^{k}\frac{\xi_i^{x-1}\mathrm{e}^{\varepsilon(\xi_i)t}}{1-\xi_i},$$
$$\varepsilon(\xi) = \frac{p}{\xi} + q\xi - 1$$

and $\mathcal{C}_R$ is a circle of radius $R$ centered at the origin with $R \gg 1$ so that all (finite) singularities of the integrand are enclosed by $\mathcal{C}_R$. Observe that $c_{m,k} = 0$ when $m > k$.

Since

$$\mathbb{P}_Y(\eta_t(x) = 1) = \sum_{m=1}^{|Y|} \mathbb{P}_Y(x_m(t) = x), \tag{4}$$

we sum the right-hand side of (3) over all $m \leqslant k$. To carry out this sum recall the $\tau$-binomial theorem

$$\sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} (-1)^j z^j \tau^{j(j-1)/2} = (1-z)(1-z\tau)\cdots(1-z\tau^{n-1}).$$

Using this a simple calculation shows

$$\sum_{m=1}^{k}(-1)^{m+1}\tau^{m(m-1)/2}\tau^{-km}\begin{bmatrix} k-1 \\ k-m \end{bmatrix} = (-1)^{k+1}\tau^{-k(k+1)/2}\prod_{j=1}^{k-1}(1-\tau^j).$$

[5] We make some changes in the notation in (5.12) of [6]. The $(p,q)$-binomial coefficient $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]$ of [6] equals $q^{k(n-k)}$-times the $\tau$-binomial coefficient $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]$ defined above. The second change is a little more subtle. The sum in (5.12) is over all finite subsets $S \subset \{1, 2, \ldots, |Y|\}$ with $|S| \geqslant m$. If $S = \{s_1, \ldots, s_k\}$ the subset $Y_S := \{y_{s_1}, \ldots, y_{s_k}\}$ and the factor $\prod_{i\in S}\xi_i^{-y_i}$ appears in the integrand of (5.12). Thus, we can equivalently sum over all finite subsets $S \subset Y$ where now the factor $\prod_{1\leqslant i\leqslant k}\xi_i^{-s_i}$ appears in the integrand. The factor $\sigma(S) = \sum_{i\in S} \mathrm{i}$ of (5.12) becomes $\sigma(S,Y)$ given above.

[6] All contour integrals are to be given a factor of $1/2\pi\mathrm{i}$.

Thus,

$$\mathbb{P}_Y(\eta_t(x)=1)=\sum_{k=1}^{|Y|}(-1)^{k+1}q^{k(k-1)/2}\tau^{-k(k+1)/2}\prod_{j=1}^{k-1}(1-\tau^j)$$
$$\times\sum_{\substack{S\subset Y\\|S|=k}}\tau^{\sigma(S,Y)}\int_{\mathcal{C}_R}\cdots\int_{\mathcal{C}_R}I(x,k,\xi)\prod_{i=1}^{k}\xi_i^{-s_i}\,\mathrm{d}^k\xi. \quad (5)$$

**Remark 1.** The above formula holds for either $|Y|$ finite or infinite. For $|Y|=N$, the integral of order $N$ in (5) is obtained from the summand $S=Y$. Since $\sigma(Y,Y)=N(N+1)/2$, we get for the coefficient of this integral

$$(-1)^{N+1}q^{N(N-1)/2}\prod_{j=1}^{N-1}(1-\tau^j)=(-1)^{N+1}\prod_{j=1}^{N-1}(q^j-p^j). \quad (6)$$

## 4. Probability for a site to be occupied by a second-class particle

As above, suppose that our initial configuration consists of a second-class particle at site 0 and first-class particles at sites in $Y$. As above, set $Y'=Y\cup\{0\}$. The process $\zeta_t$ has initially its particles at sites in $Y'$. We apply formula (5) to the initial configurations $Y'$ and $Y$ and by (1) we subtract to obtain $\mathbb{P}_Y(X(t)=x)$. If $|Y'|=N$ there is one $N$-dimensional integral that comes from the expansion of $\mathbb{P}_{Y'}(\zeta_t(x)=1)$ when $S=Y'$. The coefficient of the integral of highest order equals (6).

We now consider the special case of *step initial condition*, that is, $Y=\mathbb{Z}^+$, and use corollary (5.13) of [6] to obtain a more compact expression for $\mathbb{P}_{\mathbb{Z}^+}(x_m(t)=x)$. To find $\mathbb{P}_{\mathbb{Z}^+}(\eta_t(x)=1)$ we again apply (4) but use (5.13) of [6]. As above we interchange the sums over $k$ and $m$, use the $\tau$-binomial theorem ([4], page 26), to conclude

$$\mathbb{P}_{\mathbb{Z}^+}(\eta_t(x)=1)=-\sum_{k\geqslant 1}\frac{q^{k^2}}{k!}\prod_{j=1}^{k-1}(1-\tau^j)\int_{\mathcal{C}_R}\cdots\int_{\mathcal{C}_R}\tilde{J}_k(x,\xi)\,\mathrm{d}\xi_1\cdots\mathrm{d}\xi_k, \quad (7)$$

where

$$\tilde{J}_k(x,\xi)=\prod_{i\neq j}\frac{\xi_j-\xi_i}{p+q\xi_i\xi_j-\xi_i}\left(1-\prod_i\xi_i\right)\prod_i\frac{\xi_i^{x-1}\,\mathrm{e}^{\varepsilon(\xi_i)t}}{(1-\xi_i)(q\xi_i-p)}.$$

We can get the corresponding formula for $Y'=\mathbb{Z}^+\cup\{0\}$ by observing that there is a one–one correspondence between subsets $S'\subset Y'$ and subsets $S\subset Y$ given by $S=S'+1$. Then $\sigma(S',Y')=\sigma(S,Y)$ and, with obvious notation, $\prod\xi_i^{-s_i'}=\prod\xi_i\cdot\prod\xi_i^{-s_i}$. It follows that for the difference $\mathbb{P}_{Y'}(\zeta_t(x)=1)-\mathbb{P}_Y(\eta_t(x)=1)$ we multiply the integrand $\tilde{J}_k(x,\xi)$ in (7) by $\prod\xi_i-1$.

Thus,

$$\mathbb{P}_{\mathbb{Z}^+}(X(t)=x)=\sum_{k\geqslant 1}\frac{q^{k^2}}{k!}\prod_{j=1}^{k-1}(1-\tau^j)\int_{\mathcal{C}_R}\cdots\int_{\mathcal{C}_R}\tilde{\tilde{J}}_k(x,\xi)\,\mathrm{d}\xi_1\cdots\mathrm{d}\xi_k, \quad (8)$$

where

$$\tilde{\tilde{J}}_k(x,\xi)=\prod_{i\neq j}\frac{\xi_j-\xi_i}{p+q\xi_i\xi_j-\xi_i}\left(1-\prod_i\xi_i\right)^2\prod_i\frac{\xi_i^{x-1}\,\mathrm{e}^{\varepsilon(\xi_i)t}}{(1-\xi_i)(q\xi_i-p)}.$$

From this it follows that the distribution function is (on $\mathcal{C}_R$, $|\xi^{-1}| \ll 1$)

$$\mathbb{P}_{\mathbb{Z}^+}(X(t) \leqslant x) = \sum_{k\geqslant 1} \frac{q^{k^2}}{k!} \prod_{j=1}^{k-1}(1-\tau^j) \int_{\mathcal{C}_R} \cdots \int_{\mathcal{C}_R} J_k(x,\xi)\, \mathrm{d}\xi_1 \cdots \mathrm{d}\xi_k, \tag{9}$$

where

$$J_k(x,\xi) = \prod_{i\neq j} \frac{\xi_j - \xi_i}{p + q\xi_i\xi_j - \xi_i} \left(\prod_i \xi_i - 1\right) \prod_i \frac{\xi_i^x\, \mathrm{e}^{\varepsilon(\xi_i)t}}{(1-\xi_i)(q\xi_i - p)}.$$

Since

$$\frac{1}{p+q\xi\xi' - \xi} = \frac{1}{\xi(\xi'-1)} + \mathrm{O}(\tau), \qquad \tau \to 0,$$

the TASEP limit of $J_k(x,\xi)$ is

$$J_k^{\mathrm{TASEP}}(x,\xi) := \lim_{\tau\to 0} J_k(x,\xi) = \prod_{i\neq j}(\xi_j - \xi_i) \left(\prod \xi_i - 1\right) \prod_i \frac{\xi_i^x\, \mathrm{e}^{\varepsilon(\xi_i)t}}{(\xi_i(1-\xi_i))^k},$$

where now $\varepsilon(\xi) = \xi - 1$, and hence,

$$\lim_{\tau\to 0} \mathbb{P}_{\mathbb{Z}^+}(X(t) \leqslant x) = \sum_{k\geqslant 1} \frac{1}{k!} \int_{\mathcal{C}_R} \cdots \int_{\mathcal{C}_R} J_k^{\mathrm{TASEP}}(x,\xi)\, \mathrm{d}\xi_1 \cdots \mathrm{d}\xi_k. \tag{10}$$

Expression (9) for the distribution function can be simplified somewhat. Define the kernel

$$K_{x,t}(\xi,\xi') = q\, \frac{(\xi')^x \mathrm{e}^{\varepsilon(\xi')t}}{p + q\xi\xi' - \xi},$$

and the associated operator $K_{x,t}$ on $L^2(\mathcal{C}_R)$ by

$$f(\xi) \longrightarrow \int_{\mathcal{C}_R} K_{x,t}(\xi,\xi') f(\xi')\, \mathrm{d}\xi', \quad \xi \in \mathcal{C}_R.$$

Then using the identity [7]

$$\det\left(\frac{1}{p+q\xi_i\xi_j - \xi_i}\right)_{1\leqslant i,j\leqslant k} = (-1)^k (pq)^{k(k-1)/2} \prod_{i\neq j} \frac{\xi_j - \xi_i}{p + q\xi_i\xi_j - \xi_i} \prod_i \frac{1}{(1-\xi_i)(q\xi_i - p)}$$

we have

$$\begin{aligned}
\mathbb{P}_{\mathbb{Z}^+}(X(t) \leqslant x) &= \sum_{k\geqslant 1} \tau^{-k(k-1)/2} \prod_{j=1}^{k-1}(1-\tau^j) \\
&\quad \times \frac{(-1)^k}{k!} \int_{\mathcal{C}_R} \cdots \int_{\mathcal{C}_R} \left[\det(K_{x+1,t}(\xi_i,\xi_j))_{1\leqslant i,j\leqslant k} - \det(K_{x,t}(\xi_i,\xi_j))_{1\leqslant i,j\leqslant k}\right] \\
&= \sum_{k\geqslant 1} \tau^{-k(k-1)/2} \prod_{j=1}^{k-1}(1-\tau^j) \int_{\mathcal{C}_R} \frac{1}{\lambda^{k+1}} [\det(I - \lambda K_{x+1,t}) - \det(I - \lambda K_{x,t})]\, \mathrm{d}\lambda,
\end{aligned} \tag{11}$$

where $\det(I - \lambda K_{x,t})$ is the Fredholm determinant and the last line follows from the Fredholm expansion.

**Remark 2.**

(i) One cannot interchange the sum and the integration in (11) as was possible in an analogous calculation in [7]. This is the case even though (9) converges absolutely for all $0 \leqslant \tau \leqslant 1$

(recall one may take $R \gg 1$). Thus, we do not have a representation of $\mathbb{P}_{\mathbb{Z}^+}(X(t) \leqslant x)$ as a single integral whose integrand involves the above Fredholm determinants as was the case in [7].

(ii) ASEP with first- and second-class particles is integrable in the sense that the Yang–Baxter equations are satisfied [1]. Using this integrable structure, it is possible to compute directly, i.e. without using the basic lemma (1), $\mathbb{P}_{\mathbb{Z}^+}(X(t) = x)$ using methods similar to that of [6]. We have carried this out to the extent that (6) was computed by this approach. However, this route is much more involved than the one presented here.

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